Study Guide 1

Solutions

1. Compute the indefinite integrals below.

(a) $\displaystyle\int 3x^3 - 4x^2 + 3x + 2\,dx = \frac{3}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2 + 2x + C$

(b) $\displaystyle\int 4\sqrt{t} + \frac{2}{\sqrt[3]{t}}\,dt = \int 4t^{1/2} + 2t^{-1/3}\,dt = \frac{4}{3/2}t^{3/2} + \frac{2}{2/3}t^{2/3} + C = \frac{8}{3}t^{3/2} + 3t^{2/3} + C$

(c) $\displaystyle\int \frac{3x^2 + 2x - 1}{4x^3}\,dx = \int \frac{3}{4}x^{-1} + \frac{1}{2}x^{-2} - \frac{1}{4}x^{-3}\,dx = \frac{3}{4}\ln|x| - \frac{1}{2}x^{-1} + \frac{1}{8}x^{-2} + C$

(d) $\displaystyle\int \frac{3x\,dx}{\sqrt[3]{x^2+1}} = \ldots$

(*) Substitute $u = x^2 + 1 \implies du = 2x\,dx \implies 3x\,dx = \frac{3}{2}\,du$

$$\int \frac{3x\,dx}{\sqrt[3]{x^2+1}} = \int \frac{3/2\,du}{\sqrt[3]{u}} = \frac{3}{2}\int u^{-1/3}\,du = \frac{3}{2}\cdot\frac{u^{2/3}}{2/3} + C$$
$$= \frac{9}{4}u^{2/3} + C = \frac{9}{4}\left(x^2+1\right)^{2/3} + C$$

(e) $\displaystyle\int (x^2 + 2x)(x^3 + 3x^2 - 1)^3\,dx = \ldots$

(*) Substitute $u = x^3 + 3x^2 - 1 \implies du = (3x^2 + 6x)\,dx \implies (x^2 + 2x)\,dx = \frac{1}{3}\,du$

$$\int (x^2 + 2x)(x^3 + 3x^2 - 1)^3\,dx = \frac{1}{3}\int u^3\,du = \frac{1}{12}u^4 + C = \frac{1}{12}\left(x^3 + 3x^2 - 1\right)^4 + C$$

(f) $\displaystyle\int (x^2+1)(x+3)^2\,dx = \int (x^2+1)(x^2+6x+9)\,dx = \int x^4 + 6x^3 + 10x^2 + 6x + 9\,dx$

$$= \frac{1}{5}x^5 + \frac{3}{2}x^4 + \frac{10}{3}x^3 + 3x^2 + 9x + C$$

(g) $\displaystyle\int 1000e^{-0.05t}\,dt = \ldots$

(*) Substitute $u = -0.05t \implies du = -0.05\,dy \implies dt = -20\,du$

$$\int 1000e^{-0.05t}\,dt = -20,000\int e^u\,du = -20,000e^u + C = -20,000e^{-0.05t} + C$$

(h) $\displaystyle\int \frac{t^2+5}{3t+1}\,dt = \ldots$

(*) This one can be done using *long division* (of polynomials)

$$\frac{t^2+5}{3t+1} = \frac{1}{3}t - \frac{1}{9} + \frac{46/9}{3t+1}$$

to simplify the integrand before integration. The integration will require substituting $u = 3t + 1$. Alternatively, you can make this substitution in the original integral. This also requires solving for t (in the numerator):

$$u = 3t + 1 \implies t = \frac{1}{3}(u - 1) \text{ and } u = 3t + 1 \implies du = 3\,dt \implies dt = \frac{1}{3}\,du.$$

Using the second approach, we see that

$$\begin{aligned}\int \frac{t^2+5}{3t+1}\,dt &= \frac{1}{3}\int \frac{\left(\frac{1}{3}(u-1)\right)^2+5}{u}\,du = \frac{1}{3}\int \frac{\frac{1}{9}u^2 - \frac{2}{9}u + \frac{46}{9}}{u}\,du \\ &= \frac{1}{27}\int u - 2 + \frac{46}{u}\,du = \frac{1}{54}u^2 - \frac{2}{27}u + \frac{46}{27}\ln|u| + C \\ &= \frac{1}{54}(3t+1)^2 - \frac{2}{27}(3t+1) + \frac{46}{27}\ln|3t+1| + C\end{aligned}$$

(*) This can be further simplified, if desired:

$$\begin{aligned}&= \frac{1}{54}(9t^2+6t+1) - \frac{2}{9}t - \frac{2}{27} + \frac{46}{27}\ln|3t+1| + C \\ &= \frac{1}{6}t^2 - \frac{1}{9}t + \frac{46}{27}\ln|3t+1| + C,\end{aligned}$$

which is the answer your would get if you took the long-division approach. Note that the constants $2/27$ and $1/54$ were ‘absorbed’ by the constant of integrations C.

2. Find the function $y = f(x)$, given that $y' = x - \frac{1}{x}$, and $f(1) = 3$.

First, we integrate

$$\int y'\,dx = \int x - \frac{1}{x}\,dx = \frac{x^2}{2} - \ln|x| + C.$$

I.e., $f(x) = \frac{x^2}{2} - \ln|x| + C$, and we use the initial value to solve for C:

$$3 = f(1) = \frac{1^2}{2} - \ln 1 + C = \frac{1}{2} + C \implies C = 3 - \frac{1}{2} = \frac{5}{2}.$$

So, $f(x) = \frac{x^2}{2} - \ln|x| + \frac{5}{2}$.

3. Find the function $y = g(x)$, given that $y'' = x^2 - 1$, $g'(1) = 2$ and $g(1) = 2$.

First solve one initial value problem to find y', by integrating y'':

$$y' = \int y''\,dx = \int x^2 - 1\,dx = \frac{x^3}{3} - x + C_1.$$

Next, solve for C_1 using the initial value for y':

$$2 = y'(1) = \frac{1^3}{3} - 1 + C_1 \implies C_1 = 2 + 1 - \frac{1}{3} = \frac{8}{3}.$$

So, $y' = \dfrac{x^3}{3} - x + \dfrac{8}{3}$, and now we repeat the process to find $y = g(x)$.

$$g(x) = \int y'\,dx = \int \frac{x^3}{3} - x + \frac{8}{3}\,dx = \frac{x^4}{12} - \frac{x^2}{2} + \frac{8x}{3} + C_2.$$

Use the initial data for $g(x)$ to solve for C_2:

$$2 = g(1) = \frac{1}{12} - \frac{1}{2} + \frac{8}{3} + C_2 \implies C_2 = -\frac{1}{4},$$

giving the final solution $g(x) = \dfrac{x^4}{12} - \dfrac{x^2}{2} + \dfrac{8x}{3} - \dfrac{1}{4}$.

4. The marginal revenue function for a firm is

$$\frac{dr}{dq} = 200 - q^{2/3}.$$

Find the firm's demand function.

The demand function $p = f(q)$ is found by dividing the revenue function, $r(q)$, by q, i.e., $p = r/q$. The revenue function is found by solving the initial value problem, $r' = 200 - q^{2/3}, \quad r(0) = 0$. First,

$$r = \int 200 - q^{2/3}\,dq = 200q - \frac{3}{5}q^{5/3} + C.$$

Next, the initial value $r(0) = 0$ implies that $C = 0$, so $r = 200q - \frac{3}{5}q^{5/3}$, and the demand function is

$$p = \frac{r}{q} = 200 - \frac{3}{5}q^{2/3}.$$

5. A firm's fixed cost is \$12000, and their marginal cost function is

$$\frac{dc}{dq} = (q + 1000)^{1/3} + 50.$$

Find the firm's cost function.

Another initial value problem. First,

$$c = \int (q + 1000)^{1/3} + 50\,dq = \frac{3}{4}(q + 1000)^{4/3} + 50q + K,$$

(using the substitution $u = q + 1000$ to compute the integral). The initial values is given by $c(0) = 12000$ (because fixed cost $= c(0)$), which we use to solve for the constant of integration, K:

$$12000 = c(0) = \frac{3}{4}1000^{4/3} + K = 7500 + K \implies K = 4500.$$

So the cost function is $c = \dfrac{3}{4}(1000 + q)^{4/3} + 50q + 4500$.

(*) The next two problems are solved using the same idea. If $y = f(x) + C$, then even if we don't know C, we can compute Δy — the change in the value of y — using the formula

$$\Delta y = y(x_2) - y(x_1) = f(x_2) + C - (f(x_1) + C) = f(x_2) - f(x_1).$$

I.e., the change in y does not depend on the constant. This is useful when we know the derivative y, but not y itself. This is essentially the same as computing a *definite* integral, which we will learn next.

6. A firm's marginal revenue and marginal cost functions are

$$\frac{dr}{dq} = 100 - \sqrt{3q+10} \quad \text{and} \quad \frac{dc}{dq} = 0.2q + 65,$$

respectively. How will the firm's **profit** change if output is increased from $q = 30$ to $q = 53$?

The profit function is given by $\pi = r - c$, so the derivative of the profit function (in this problem) is given by

$$\frac{d\pi}{dq} = \frac{dr}{dq} - \frac{dc}{dq} = 100 - \sqrt{3q+10} - (0.2q + 65) = 35 - 0.2q - \sqrt{3q+10}.$$

This means that the profit function is given by

$$\begin{aligned}\pi &= \int 35 - 0.2q - \sqrt{3q+10}\, dq = \int 35 - 0.2q\, dq - \int \sqrt{3q+10}\, dq \\ &= 35q - 0.1q^2 - \frac{1}{3}\int u^{1/2}\, du = 35q - 0.1q^2 - \frac{1}{3}\cdot\frac{u^{3/2}}{3/2} + C \\ &= 35q - 0.1q^2 - \frac{2}{9}(3q+10)^{3/2} + C,\end{aligned}$$

using the substitution $u = 3q + 10 \implies du = \frac{1}{3}\, dq$ in the last integral in first line.

It follows that the change in the firm's profit is

$$\begin{aligned}\Delta\pi &= \pi(53) - \pi(30) \\ &= \left(35\cdot 53 - 0.1\cdot 2809 - \frac{2}{9}\cdot 169^{3/2}\right) - \left(35\cdot 30 - 0.1\cdot 900 - \frac{2}{9}\cdot 100^{3/2}\right) \\ &= 1085.8\overline{777} - 737.7\overline{777} = 348.10\end{aligned}$$

7. The marginal propensity to consume of a small nation is given by

$$\frac{dC}{dY} = \frac{9Y+10}{10Y+1},$$

where consumption C and national income Y are both measured in billions of dollars. Find the total change in national consumption and saving, if income increases from \$10 billion to \$15 billion.

Using the substitution

$$u = 10Y + 1 \implies Y = \frac{1}{10}(u-1) \text{ and } dY = \frac{1}{10}\,du$$

to compute the integral, we see that the nation's consumption function is

$$\begin{aligned} C &= \int \frac{9Y+10}{10Y+1}\,dY = \frac{1}{10}\int \frac{\frac{9}{10}(u-1)+10}{u}\,du \\ &= \frac{1}{100}\int \frac{9u+91}{u}\,du = \frac{1}{100}\int 9 + \frac{91}{u}\,du \\ &= \frac{9}{100}u + \frac{91}{100}\ln|u| + K \\ &= \frac{9}{100}(10Y+1) + \frac{91}{100}\ln|10Y+1| + K \\ &= 0.9Y + 0.91\ln|10Y+1| + K. \end{aligned}$$

Observe that the constant 9/100 in the fourth line was 'absorbed' by the constant of integration K. It follows that the change in consumption is

$$\Delta C = C(15) - C(10) = 0.9 \cdot 15 + 0.91\ln|151| - 0.9 \cdot 10 - 0.91\ln|101| \approx 4.866,$$

and therefore the change in saving is

$$\Delta S = \Delta Y - \Delta C \approx 5 - 4.866 = 0.134.$$

In words, if income increases from \$10 billion to \$15 billion, consumption will increase by about \$4.866 billion and saving will increase by about \$134 million.

(*) I also used the fact from economics that $Y = C + S$, so $S = Y - C$ and therefore $\Delta S = \Delta Y - \Delta C$.