Study Guide 2

Solutions

1. Compute the definite integrals below.

a. $$\int_0^2 2x^3 + x^2 - 5x + 2\,dx = \frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{5}{2}x^2 + 2x\Big|_0^2$$
$$= \left(8 + \frac{8}{3} - 10 + 4\right) - (0 + 0 - 0 + 0) = \frac{14}{3}$$

b. $$\int_1^8 2\sqrt[3]{t} + \frac{3}{\sqrt[3]{t^2}}\,dt = \int_1^8 2t^{1/3} + 3t^{-2/3}\,dt = \frac{3}{2}t^{4/3} + 9t^{1/3}\Big|_1^8$$
$$= \left(\frac{3}{2}\cdot 16 + 9\cdot 2\right) - \left(\frac{3}{2} + 9\right) = 31.5$$

c. *Use the substitution* $u = 4x + 1 \implies dx = \frac{1}{4}\,du$. *This also causes the limits of integration to change:* $x = 0 \implies u = 1$ *and* $x = 4 \implies u = 17$...
$$\int_0^4 \frac{5}{4x+1}\,dx = \frac{5}{4}\int_1^{17} \frac{1}{u}\,du = \frac{5}{4}\ln u\Big|_1^{17} = \frac{5}{4}\ln 17 - \frac{5}{4}\ln 1 = \frac{5\ln 17}{4} \approx 3.5415$$

d. *Use the substitution* $u = -0.04t \implies dt = -25\,du$. *This also causes the limits of integration to change:* $t = 0 \implies u = 0$ *and* $t = 20 \implies u = -0.8$...
$$\int_0^{20} 500e^{-0.04t}\,dt = -12500\int_0^{-0.8} e^u\,du = -12500e^u\Big|_0^{-0.8}$$
$$= -12500e^{-0.8} - (-12500e^0)$$
$$= 12500(1 - e^{-0.8}) \approx 6883.388$$

e. *Use the substitution* $u = t^2 + 9 \implies du = 2t\,dt \implies t\,dt = \frac{1}{2}\,du$. *This also changes the limits of integration:* $t = 0 \implies u = 9$ *and* $t = 4 \implies u = 25$...
$$\int_0^4 3t\sqrt{t^2+9}\,dt = \frac{3}{2}\int_9^{25} u^{1/2}\,du = \frac{\cancel{3}}{\cancel{2}}\cdot\frac{u^{3/2}}{\cancel{3/2}}\Big|_9^{25}\ 25^{3/2} - 9^{3/2} = 125 - 27 = 98$$

2. Find the area of the region bounded by the graphs $y = 2\sqrt{x}$ and $y = 1 - 2x$, and the lines $x = 1$ and $x = 4$.

If $1 \le x \le 4$, *then* $2\sqrt{x} > 1 - 2x$, *as illustrated in Figure 1 below, and the region whose area we want to calculate is the region* R *in this figure bounded by the red line segments. This means that*

$$\text{area}(R) = \int_1^4 2\sqrt{x} - (1 - 2x)\,dx$$
$$= \int_1^4 2x + 2x^{1/2} - 1\,dx$$
$$= x^2 + \frac{4}{3}x^{3/2} - x\Big|_1^4 = \left(16 + \frac{32}{3} - 4\right) - \left(1 + \frac{4}{3} - 1\right) = \frac{64}{3}$$

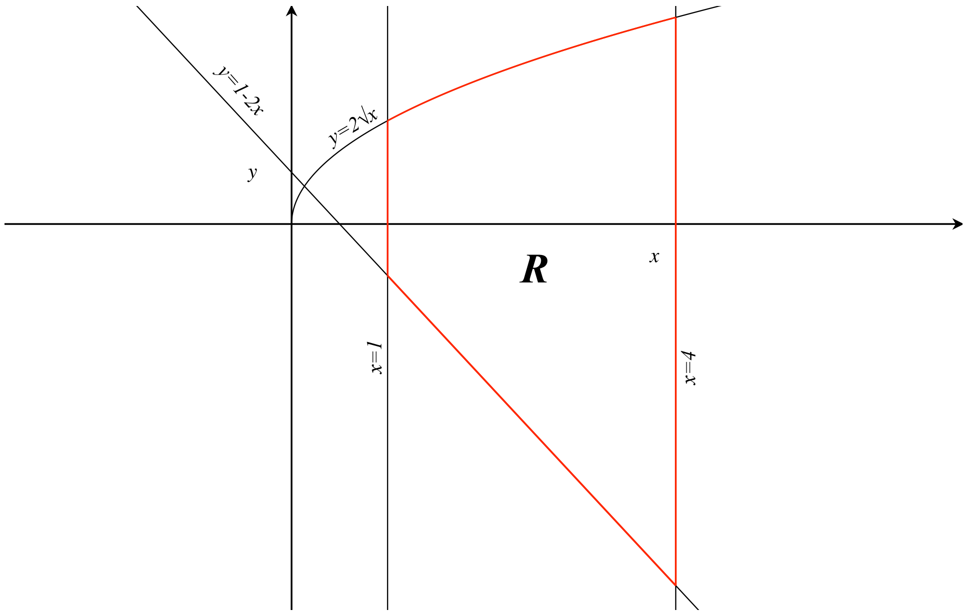

Figure 1: The region R in problem 2.

3. What are the *Producers' surplus* and *Consumers' surplus* for the market with supply function

$$p = 0.05q^2 + 3q + 5$$

and demand function

$$p = 100 - 0.75q.$$

First we find the market equilibrium price and demand:

$$\textit{supply} = \textit{demand} \implies 0.05q^2 + 3q + 5 = 100 - 0.75q \implies 0.05q^2 + 3.75q - 95 = 0$$

From the quadratic formula, we find that

$$q = \frac{-3.75 \pm \sqrt{3.75^2 - 4 \cdot 0.2 \cdot (-95)}}{0.1} = \begin{cases} \dfrac{-3.75 + 5.75}{0.1} & = \quad 20 \\ & \textit{or} \\ \dfrac{-3.75 - 5.75}{0.1} & = \quad -95 \end{cases}$$

So, the equilibrium demand is $q^* = 20$ *(because demand must be positive) and the equilibrium price is* $p^* = 100 - 0.75q^* = 85$.

Next we compute the consumers' and producers' surplus:

$$CS = \int_0^{q^*} (\textit{demand} - p^*)\, dq = \int_0^{20} 100 - 0.75q - 85\, dq = 15q - \frac{3}{8}q^2 \Big|_0^{20} = 150$$

and

$$PS = \int_0^{q^*} (p^* - \textit{supply})\, dq = \int_0^{20} 85 - (0.05q^2 + 3q + 5)\, dq = 80q - \frac{3}{2}q^2 - \frac{1}{60}q^3 \Big|_0^{20} \approx 866.67$$

4. Find the average value of the function $f(x) = \dfrac{x^4 - 1}{x^2}$ on the interval $[1, 3]$.

The average value of a function on an interval is equal to the definite integral of the function on the interval, divided by the length of the interval, i.e.,

$$avg(f) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

In this case, we get

$$avg(f) = \frac{1}{3-1} \int_1^3 \frac{x^4 - 1}{x^2}\, dx = \frac{1}{2} \int_1^3 x^2 - x^{-2}\, dx = \frac{1}{6}x^3 + \frac{1}{2}x^{-1} \Big|_1^3 = \frac{28}{6} - \frac{4}{6} = 4$$

5. Find the Gini coefficient of inequality for the nation with income distribution curve

$$y = 0.5x^3 + 0.3x^2 + 0.2x,$$

where $y \cdot 100\%$ is the percentage of national income earned by the poorest $x \cdot 100\%$ of the population.

Recall (see exercise 59 in section 14.9 and your notes) that the Gini coefficient of inequality for a nation whose income distribution curve is given by $y = f(x)$, is given by

$$\gamma = \frac{\displaystyle\int_0^1 x - f(x)\, dx}{\displaystyle\int_0^1 x\, dx} = 2 \int_0^1 x - f(x)\, dx = 1 - 2 \int_0^1 f(x)\, dx.$$

In this problem, $f(x) = 0.5x^3 + 0.3x^2 + 0.2x$, so

$$\gamma = 1 - 2 \int_0^1 0.5x^3 + 0.3x^2 + 0.2x\, dx = 1 - 2 \left[\frac{x^4}{8} + \frac{x^3}{10} + \frac{x^2}{10} \Big|_0^1 \right] = 1 - 2\left(\frac{13}{40} - 0\right) = 0.35$$

6. The marginal propensity to *save* of a small nation is given by

$$\frac{dS}{dY} = \frac{Y + 5}{9Y + 10},$$

where *savings* S and national income Y are both measured in billions of dollars. Express the total change in national savings when income increases from \$10 billion to \$15 billion as a definite integral, and find its value. What is the total change in national consumption?

From the fundamental theorem of calculus, it follows that

$$\Delta S = S(15) - S(10) = \int_{10}^{15} \frac{dS}{dY}\, dY$$

so

$$\begin{aligned}
\Delta S &= \int_{10}^{15} \frac{Y+5}{9Y+10}\, dY \\
&= \frac{1}{9}\int_{100}^{145} \frac{\frac{1}{9}(u-10)+5}{u}\, du \\
&= \frac{1}{81}\int_{100}^{145} 1 + \frac{35}{u}\, du \\
&= \frac{1}{81}(u + 35\ln|u|)\Big|_{100}^{145} \\
&= \left(\frac{145}{81} + \frac{35}{81}\ln 145\right) - \left(\frac{100}{81} + \frac{35}{81}\ln 100\right) \approx 0.7161
\end{aligned}$$

The total change in consumption is given by the identity $C = Y - S$, so

$$\Delta C = \Delta Y - \Delta S \approx 5 - 0.7161 = 4.2839.$$