Solutions

1. Use the table of integral formulas in Appendix B in the textbook to help compute the integrals below.

a. $$\int \frac{4\,dx}{5x\sqrt{x^2+9}} = \frac{4}{5}\cdot\frac{1}{3}\ln\left|\frac{\sqrt{x^2+9}-3}{x}\right| + C = \frac{4}{15}\ln\left|\frac{\sqrt{x^2+9}-3}{x}\right| + C$$

Formula #28, with $a = 3$.

b. $$\int \frac{2e^{2x}\,dx}{\sqrt{9+4e^x}} = 2\int \frac{e^x e^x\,dx}{\sqrt{9+4e^x}} = 2\int \frac{u\,du}{\sqrt{9+4u}}$$
$$= 2\cdot\frac{2(4u-18)\sqrt{9+4u}}{48} + C = \frac{(2e^x-9)\sqrt{9+4e^x}}{6} + C$$

First, note that $e^{2x} = e^x e^x$ and substitute $u = e^x$, $du = e^x\,dx$, then use formula #15 with $a = 9$ and $b = 4$.

c. $$\int_0^{10} 200t^2e^{-0.06t}\,dt = 200\left(\left.\frac{t^2e^{-0.06t}}{-0.06}\right|_0^{10} - \frac{2}{-0.06}\int_0^{10} te^{-0.06t}\,dt\right)$$
$$= 200\left(\left(-\frac{100e^{-0.6}}{0.06} - 0\right) + \frac{2}{0.06}\cdot\left.\frac{e^{-0.06t}}{0.06^2}(-0.06t-1)\right|_0^{10}\right)$$
$$= 200\left(-\frac{100e^{-0.6}}{0.06} - \frac{3.2}{0.06^3}e^{-0.6} + \frac{2}{0.06^3}\right) \approx 42806.0884$$

Formula #39 with $n = 2$ and $a = -0.06$, followed by formula #38 (with $a = -0.06$ again).

d. $$\int_0^3 \frac{2\,dv}{\sqrt{v^2+16}} = 2\ln\left|v+\sqrt{v^2+16}\right|\Big|_0^3 = 2(\ln(3+5) - \ln(0+4)) = 2\ln 2$$

Formula #27, with $a = 4$.

e. $$\int 5x^3\ln x\,dx = \frac{5x^4\ln x}{4} - \frac{5x^4}{16} + C$$

Formula #42, with $n = 3$.

f. $$\int_0^2 \frac{3+5x}{2+7x}\,dx = 3\int_0^2 \frac{1}{2+7x}\,dx + 5\int_0^2 \frac{x}{2+7x}\,dx = \left(\left.\frac{3}{7}\ln|2+7x|\right|_0^2\right) + \left(\left.\frac{5x}{7} - \frac{10}{49}\ln|2+7x|\right|_0^2\right)$$
$$= \frac{3}{7}(\ln 16 - \ln 2) + \left(\frac{10}{7} - \frac{10}{49}\ln 16\right) - \left(0 - \frac{10}{49}\ln 2\right) = \frac{10}{7} + \frac{11}{49}\ln 8$$

Formulas #2 and #3, with $a = 2$ and $b = 7$.

2. Compute the present value of the continuous annuity that pays at the continuous rate $f(t) = 250t$ for $T = 20$ years, where the constant interest rate is $r = 4.75\%$.

See section 15.3 in the text, and formula #38.

$$\begin{aligned}
\text{Present Value} &= \int_0^T f(t)e^{-rt}\,dt \\
&= \int_0^{20} 250te^{-0.0475t}\,dt \\
&= \frac{250e^{-0.0475t}}{(0.0475)^2}(-0.0475t - 1)\Big|_0^{20} \approx 27241.55.
\end{aligned}$$

3. Let $y = f(x)$ satisfy (i) $\dfrac{dy}{dx} = 3xy^2$ and (ii) $y(1) = 2$. Find the function $f(x)$.

Separate: $\dfrac{dy}{y^2} = 3x\,dx.$

Integrate: $\displaystyle\int \frac{dy}{y^2} = \int 3x\,dx \quad\Longrightarrow\quad -\frac{1}{y} = \frac{3x^2}{2} + C.$ (This is the *implicit* solution.)

Solve for y: $y = \dfrac{2}{C - 3x^2}.$

Solve for C: $y(1) = 2 \implies 2 = \dfrac{2}{C-3} \implies C - 3 = 1 \implies C = 4.$

Solution: $y = \dfrac{2}{4 - 3x^2}.$

4. The dead body of an eccentric socialite is found in a Las Vegas motel room. At 10:00 am, her body temperature was measured to be 92.4°F. Her body was left in the room for an hour and at 11:00 am her body temperature was 90.5°F. The room itself was kept at a constant temperature of 72°F.

Use *Newton's law of cooling* to estimate the time of the socialite's death.

(*) See Example 3 in section 15.6.

Newton's law of cooling (or heating), says that the the rate of change in the temperature of a body immersed in a medium of constant ambient temperature is proportional to the difference between the temperature of the body T and the ambient temperature A. If $T(t)$ is the temperature of the body at time t, then Newton's law can be expressed as

$$\frac{dT}{dt} = k(T - A),$$

where k is the unknown constant of proportionality.† To solve this differential equation,

†The constant k depends on the physical properties of the body, in particular its heat conductance.

we separate, integrate and solve for T:

$$\begin{aligned}
\text{Separate}: &\ \frac{dT}{T-A} = k\,dt \\
\text{Integrate}: &\ \int \frac{dT}{T-A} = \int k\,dt \implies \ln|T-A| = kt + C \\
\text{Solve for T}: &\ |T-A| = e^{kt+C} = e^C \cdot e^{kt} \implies T - A = \pm e^C \cdot e^{kt} \implies T = A + \alpha e^{kt},
\end{aligned}$$

where $\alpha = \pm e^C$.

Next, we use the given data. First, the constant ambient temperature is $A = 72°$F, so $T = 72 + \alpha e^{kt}$. Second, we'll let $t = 0$ correspond to 10:00 am, so

$$92.4 = T(0) = 72 + \alpha e^{k\cdot 0} = 72 + \alpha \implies \alpha = 92.4 - 72 = 20.4.$$

Next, measuring time in hours, 11:00 am corresponds to $t = 1$, so

$$90.5 = T(1) = 72 + 20.4e^{k\cdot 1} = 72 + 20.4e^k \implies e^k = \frac{90.5 - 72}{20.4} \implies k = \ln\frac{18.5}{20.4}$$

Finally, assuming that the socialite's temperature was normal (98.6°F) at her time of death (t_d) , we find that

$$98.6 = T(t_d) = 72{+}20.4e^{kt_d} \implies e^{kt_d} = \frac{26.6}{20.4} \implies kt_d = \ln\frac{26.6}{20.4} \implies t_d = \frac{\ln\frac{26.6}{20.4}}{\ln\frac{18.5}{20.4}} \approx -2.714.$$

This means that the socialite died 2.714 hours *before* 10:00 am—at roughly 7:17 am.

5. The population of a tropical island grows at a rate that is proportional to the *third root* ($\sqrt[3]{\ }$) of its size. In 1950, the islands population was 1728 and in 1980, the islands population was 2744. What will the islands population be in 2020?

First, translate the description of the growth rate into a differential equation. If $P(t)$ is the size of the population at time t (in years), then the description above leads to the differential equation

$$\frac{dP}{dt} = k\sqrt[3]{P},$$

where k is the (unknown) constant of proportionality.

Separate the variables: $\dfrac{dP}{P^{1/3}} = k\,dt.$

Integrate both sides: $\displaystyle\int P^{-1/3}\,dP = \int k\,dt \quad\implies\quad \frac{3}{2}P^{2/3} = kt + C.$

Solve for P: Multiplying by 2/3 gives $P^{2/3} = kt + C$, because the factor of 2/3 is absorbed by both k and C. Next, raise both sides to the power 3/2 to see that

$$P = (kt + C)^{3/2}.$$

Solve for the parameters k and C: First set $t = 0$ for the year 1950, so

$$1728 = P(0) = (0 + C)^{3/2} = C^{3/2} \quad \Longrightarrow \quad \boxed{C = 1728^{2/3} = 144}.$$

Next, the year 1980 corresponds to $t = 30$, so

$$2744 = P(3) = (30k + 144)^{3/2} \implies 30k + 144 = 2744^{2/3} = 196 \implies \boxed{k = \frac{26}{15}}.$$

Thus $P(t) = \left(\frac{26}{15}t + 144\right)^{3/2}$ and in 2020 the population will be

$$P(70) = \left(\frac{26}{15} \cdot 70 + 144\right)^{3/2} \approx 4322.$$

6. The income-elasticity of monthly demand (q) for a price-controlled good is assumed to be proportional to the natural logarithm of average monthly disposable income (y) in the market for that good. When $y = 2500$, the demand is $q = 500$ and when $y = 2000$, the demand is $q = 350$. What is the predicted monthly demand for this good if monthly disposable income decreases to $y = 1500$?

The income-elasticity of demand is $\eta = \dfrac{dq}{dy} \cdot \dfrac{y}{q}$, and if this is proportional to the natural logarithm of income, we obtain the separable differential equation

$$\frac{dq}{dy} \cdot \frac{y}{q} = k \ln y \implies \frac{dq}{q} = k \frac{\ln y}{y}\, dy,$$

with k being the unknown constant of proportionality (as usual). Integrating both sides gives

$$\int \frac{dq}{q} = k \int \frac{\ln y}{y}\, dy \implies \ln q = k\frac{(\ln y)^2}{2} + C = k(\ln y)^2 + C,$$

using the substitution $u = \ln y,\ du = \frac{1}{y}\, dy$ for the right-hand integral and absorbing the factor of $1/2$ into k.[‡] Exponentiating both sides gives

$$q = e^{k(\ln y)^2 + C} = e^C \cdot e^{k(\ln y)^2} = Ae^{k(\ln y)^2},$$

and it remains to use the given data to solve for A and k. The data $q(2500) = 500$ and $q(2000) = 350$ gives a pair of equations for A and k:

$$\left.\begin{aligned} Ae^{k(\ln 2000)^2} &= 350 \\ Ae^{k(\ln 2500)^2} &= 500 \end{aligned}\right\} \implies \frac{\cancel{A}e^{k(\ln 2000)^2}}{\cancel{A}e^{k(\ln 2500)^2}} = \frac{350}{500} \implies e^{k\left[(\ln 2000)^2 - (\ln 2500)^2\right]} = 0.7$$

Next, take the natural logarithm of both sides to find k:

$$k\left[(\ln 2000)^2 - (\ln 2500)^2\right] = \ln 0.7 \implies k = \frac{\ln 0.7}{(\ln 2000)^2 - (\ln 2500)^2} \approx 0.103625.$$

To find A, use one of the two equations:

$$Ae^{k(\ln 2000)^2} = 350 \implies A = 350e^{-k(\ln 2000)^2} \approx 0.879.$$

Finally, we can predict the demand when income decreases to $y = 1500$:

$$q(1500) = Ae^{k(\ln 1500)^2} \approx 0.879e^{0.103625(\ln 1500)^2} \approx 224.35.$$

[‡]Why did I not write $\ln|q|$ and $\ln|y|$ here?

7. Compute the indicated partial derivatives of the functions given below.

a. $f(x, y, z) = 2x^3y^2z + 3x^2y^5z^3 - 4xy^3 + yz^4$.

$f_x = 6x^2y^2z + 6xy^5z^3 - 4y^3$

$f_y = 4x^3yz + 15x^2y^4z^3 - 12xy^2 + z^4$

$f_z = 2x^3y^2 + 9x^2y^5z^2 + 4yz^3$

b. $w = \dfrac{u^2v}{u + v^2}$.

$$\frac{\partial w}{\partial u} = \frac{2uv(u + v^2) - u^2v \cdot 1}{(u + v^2)^2} = \frac{u^2v + 2uv^3}{(u + v^2)^2}$$

$$\frac{\partial w}{\partial v} = \frac{u^2(u + v^2) - u^2v(2v)}{(u + v^2)^2} = \frac{u^3 - u^2v^2}{(u + v^2)^2}$$

c. $F(x, y, z, \lambda) = 10\ln(x^2y^5z^3) - \lambda(5x + 2y + 8z)$.

$$F_x = \frac{10}{x^{\cancel{2}}\cancel{y^5z^3}} \cdot 2\cancel{xy^5z^3} - 5\lambda = \frac{20}{x} - 5\lambda$$

If you use properties of the log function to simplify F before you differentiate, the differentiation is a little easier:

$$F(x, y, z, \lambda) = 20\ln x + 50\ln y + 30\ln z - \lambda(5x + 2y + 8z) \implies F_x = \frac{20}{x} - 5\lambda$$

$F_\lambda = -(5x + 2y + 8z)$.

d. $z = 2y^2e^{x^2y}$.

$\dfrac{\partial z}{\partial x} = 2y^2e^{x^2y} \cdot 2xy = 4xy^3e^{x^2y}$ In this one, y^2 behaves like a constant factor.

$\dfrac{\partial z}{\partial y} = 4ye^{x^2y} + 2y^2e^{x^2y} \cdot x^2 = (4y + 2x^2y^2)e^{x^2y}$. We have to use the product rule here.