Partial derivatives, linear approximation and optimization

1. Find the indicated partial derivatives of the functions below.

a. $z = 3x^2 + 4xy - 5y^2 - 4x + 7y - 2$,

$z_x = 6x + 4y - 4$

$z_{yx} = z_{xy} = 4$

b. $F(u, v, w) = 60u^{2/3}v^{1/6}w^{1/2}$

$\dfrac{\partial F}{\partial u} = 40u^{-1/3}v^{1/6}w^{1/2}$

$\dfrac{\partial^2 F}{\partial w \partial u} = \dfrac{\partial^2 F}{\partial u \partial w} = 20u^{-1/3}v^{1/6}w^{-1/2}$

c. $w = x^2 z \ln(y^2 + z^3)$

$w_x = 2xz \ln(y^2 + z^3)$

$w_y = x^2 z \cdot \dfrac{2y}{y^2 + z^3} = \dfrac{2x^2yz}{y^2 + z^3}$

$w_{xx} = 2z \ln(y^2 + z^3)$

$$w_{yz} = \frac{2x^2y(y^2 + z^3) - 3z^2(2x^2yz)}{(y^2 + z^3)^2} = \frac{2x^2y^3 - 4x^2yz^3}{(y^2 + z^3)^2} \quad \left(= \frac{2x^2y(y^2 - 2z^3)}{(y^2 + z^3)^2}\right)$$

$$w_{xyz} = w_{yzx} = \frac{4xy(y^2 - 2z^3)}{(y^2 + z^3)^2}$$

d. $q(u, v) = \dfrac{u^2v - 3uv^3}{2u + 3v}$

$$\frac{\partial q}{\partial u} = \frac{(2uv - 3v^3)(2u + 3v) - 2(u^2v - 3uv^3)}{(2u + 3v)^2} = \frac{2u^2v + 6uv^2 - 9v^4}{(2u + 3v)^2}$$

$$\frac{\partial q}{\partial v} = \frac{(u^2 - 9uv^2)(2u + 3v) - 3(u^2v - 3uv^3)}{(2u + 3v)^2} = \frac{2u^3 - 18u^2v^2 - 18uv^3}{(2u + 3v)^2}$$

2. The monthly cost function for ACME Widgets is

$$C = 0.02Q_A^2 + 0.01Q_AQ_B + 0.03Q_B^2 + 35Q_A + 28Q_B + 5000,$$

where Q_A and Q_B are the monthly outputs of type A widgets and type B widgets, respectively, measured in 100s of widgets. The cost is measured in dollars.

a. Compute the marginal cost of type A widgets and the marginal cost of type B widgets, if the monthly outputs are 25000 type A widgets and 36000 type B widgets.

$\partial C/\partial Q_A = 0.04Q_A + 0.01Q_B + 35$ and $\partial C/\partial Q_B = 0.01Q_A + 0.06Q_B + 28$. Next, remembering the units, $Q_A = 25000/100 = 250$ and $Q_B = 36000/100 = 360$, so

$$\left.\frac{\partial C}{\partial Q_A}\right|_{\substack{Q_A=250\\Q_B=360}} = 48.6 \quad \text{and} \quad \left.\frac{\partial C}{\partial Q_B}\right|_{\substack{Q_A=250\\Q_B=360}} = 52.1.$$

b. Suppose that production of type A widgets is held fixed at 25000, and production of type B widgets is increased from 36000 to 36050. Use your answer to part a. to estimate the change in cost to the firm.

Approximation formula: $\Delta C \approx \left(\left. \dfrac{\partial C}{\partial Q_B} \right|_{\substack{Q_A=250 \\ Q_B=360}} \right) \cdot \Delta Q_B$, since we are assuming in this case that $\Delta Q_A = 0$. Now, $\Delta Q_B = 50/100 = 0.5$, so $\Delta C \approx (52.1)(0.5) = 26.05$.

c. Suppose that production of type A widgets is increased from 25000 to 25060, and production of type B widgets is increased from 36000 to 36040. Use your answer to part a. to estimate the change in cost to the firm.

General Approximation formula: $\Delta C \approx \left(\left. \dfrac{\partial C}{\partial Q_A} \right|_{\substack{Q_A=250 \\ Q_B=360}} \right) \cdot \Delta Q_A + \left(\left. \dfrac{\partial C}{\partial Q_B} \right|_{\substack{Q_A=250 \\ Q_B=360}} \right) \cdot \Delta Q_B.$

In this case we have $\Delta Q_A = 60/100 = 0.6$ and $\Delta Q_B = 40/100 = 0.4$, so

$$\Delta C \approx \left(\left. \frac{\partial C}{\partial Q_A} \right|_{\substack{Q_A=250 \\ Q_B=360}} \right) \cdot \Delta Q_A + \left(\left. \frac{\partial C}{\partial Q_B} \right|_{\substack{Q_A=250 \\ Q_B=360}} \right) \cdot \Delta Q_B = (48.6)(0.6) + (52.1)(0.4) = 50.$$

3. The demand function for a firm's product is given by $Q = \dfrac{30\sqrt{6Y + 5p_s}}{3p+5}$, where

- Q is the monthly demand for the firm's product, measured in 1000's of units,
- Y is the average monthly disposable income in the market for the firm's product, measured in 1000s of dollars,
- p_s is the average price of a substitute for the firm's product, measured in dollars,
- p is the price of the firm's product, also measured in dollars.

a. Find Q, Q_Y, Q_{p_s} and Q_p when the monthly income is \$2500 and the prices are $p_s = 17$ and $p = 15$. Round your (final) answers to two decimal places.

First, writing $Q = Q(p, p_s, Y)$, and remembering that Y is measured in \$1000s, we have

$$Q(15, 17, 2.5) = \frac{30\sqrt{15 + 85}}{50} = 6.$$

Next, the partial derivatives are

$$Q_Y = \frac{30}{3p+5} \cdot \frac{3}{(6Y + 5p_s)^{1/2}} = \frac{90}{(3p+5)(6Y+5p_s)^{1/2}} \implies Q_Y(15, 17, 2.5) = \frac{9}{50} = 0.18,$$

$$Q_{p_s} = \frac{30}{3p+5} \cdot \frac{5}{2(6Y + 5p_s)^{1/2}} = \frac{75}{(3p+5)(6Y+5p_s)^{1/2}} \implies Q_{p_s}(15, 17, 2.5) = \frac{3}{20} = 0.15$$

and

$$Q_p = 30\sqrt{6Y + 5p_s} \cdot (-1) \cdot \frac{3}{(3p+5)^2} = -\frac{90\sqrt{6Y+5p_s}}{(3p+5)^2} \implies Q_p(15, 17, 2.5) = -\frac{9}{25} = -0.36$$

b. Compute the *income-elasticity of demand* for the firm's product at the point in part **a.**

$$\eta_{Q/Y} = Q_Y \cdot \frac{Y}{Q} \implies \eta_{Q/Y}\Big|_{\substack{p=15 \\ p_S=17 \\ Y=2.5}} = 0.18 \cdot \frac{2.5}{6} = 0.075$$

c. Use *linear approximation* and your answer to **a.** to estimate the change in demand for the firm's product if the price of the firm's product increases to \$16 and the price of substitutes increases to \$18, but income remains fixed.

$$\Delta Q \approx Q_p \cdot \Delta p + Q_{p_s} \cdot \Delta p_s + Q_Y \cdot \Delta Y = -0.36 \cdot 1 + 0.15 \cdot 1 + 0.18 \cdot 0 = -0.21.$$

I.e., demand will *decrease* by about 210 units.

d. Use your answer to part **b.** to estimate the *percentage* change in demand for the firm's product if the average income increases to \$2600 while the prices stay the same as they were in part a.

The percentage change in income is $\%\Delta Y = \frac{0.1}{2.5} \cdot 100\% = 4\%$, so

$$\%\Delta Q \approx \eta_{Q/Y} \cdot \%\Delta Y = 0.075 \cdot 4\% = 0.3\%.$$

4. Find the critical points of the functions below.

a. $f(x,y) = 3x^2 - 12xy + 19y^2 - 2x - 4y + 5$. First order conditions:

$$\left.\begin{array}{rcrcl} f_x & = & 6x - 12y - 2 & = & 0 \\ f_y & = & -12x + 38y - 4 & = & 0 \end{array}\right\}$$

Now, $f_x = 0 \implies 6x = 12y + 2$, so $12x = 24y + 4$. Plugging this into the second equation gives

$$\implies -(24y+4) + 38y - 4 = 0 \implies 14y - 8 = 0 \implies y_0 = \frac{4}{7} \implies x_0 = \frac{31}{21}.$$

So there is one critical point, $\left(\frac{31}{21}, \frac{4}{7}\right)$.

b. $g(s,t) = s^3 + 3t^2 + 12st + 2$. First order conditions:

$$\left.\begin{array}{rcrcl} g_s & = & 3s^2 + 12t & = & 0 \\ g_t & = & 6t + 12s & = & 0 \end{array}\right\}$$

Now, $g_t = 0 \implies t = -2s$, and plugging this into the first equation gives

$$3s^2 - 24s = 0 \implies 3s(s-8) = 0 \implies \text{two solutions: } s_1 = 0 \text{ and } s_2 = 8.$$

So there are two critical points in this case, $(s_1, t_1) = (0,0)$ and $(s_2, t_2) = (8, -16)$.

c. $h(u,v) = u^3 + v^3 - 3u^2 - 3v + 5$. First order conditions:

$$\begin{array}{rcccl} h_u &=& 3u^2 - 6u &=& 0, \\ h_v &=& 3v^2 - 3 &=& 0. \end{array}$$

The first equation factors as $3u(u-2) = 0$, which has two solutions $u_1 = 0$ and $u_2 = 2$. The second equation factors as well, giving $3(v^2 - 1) = 0$, which has the two solutions $v_1 = 1$ and $v_2 = -1$.

The first equation places no restrictions on the variable v, while the second equation places no restrictions on the variable u, so the critical points of the function $h(u,v)$ are

$$(u_1, v_1) = (0,1), \quad (u_1, v_2) = (0,-1), \quad (u_2, v_1) = (2,1) \text{ and } (u_2, v_2) = (2,-1).$$

5. Use the second derivative test to classify the critical values of the functions in the previous problem.

a. Second derivative test:

$$\left.\begin{array}{rcr} f_{xx} &=& 6 \\ f_{yy} &=& 38 \\ f_{xy} &=& -12 \end{array}\right\} \implies D = 6 \cdot 38 - 144 = 84 > 0.$$

Since $D > 0$ and $f_{xx} > 0$, it follows that $f\left(\frac{31}{21}, \frac{4}{7}\right) = \frac{50}{21}$ is a relative minimum value. (In fact, since the second derivatives are all constant, this is the absolute minimum value.)

b. Second derivative test:

$$\left.\begin{array}{rcr} g_{ss} &=& 6s \\ g_{tt} &=& 6 \\ g_{st} &=& 12 \end{array}\right\} \implies D(s,t) = 36s - 144.$$

Since $D(0,0) = -144 < 0$, the first critical point yields a **saddle point** on the graph of $g(s,t)$, i.e., $g(0,0) = 2$ is neither max nor min. Since $D(8,-16) = 144 > 0$ and $g_{ss}(8,-16) = 48 > 0$, it follows that $g(8,-16) = -254$, is a relative minimum value.

Can you show that $g(8,-16) = -254$ is **not** the absolute minimum?

c. Second derivative test:

$$\left.\begin{array}{rcc} h_{uu} &=& 6u - 6 \\ h_{vv} &=& 6v \\ h_{uv} &=& 0 \end{array}\right\} \implies D(u,v) = 36v(u-1).$$

Evaluating the discriminant at the four critical points we find that

i. $D(0,1) = -36 < 0$, so $h(0,1) = 3$ is neither a local minimum value nor a local maximum value;

ii. $D(0,-1) = 36 > 0$ and $h_{uu}(0,-1) = -6 < 0$, so $h(0,-1) = 7$ is a local maximum value;

iii. $D(2,1) = 36 > 0$ and $h_{uu}(2,1) = 6 > 0$, so $h(2,1) = -1$ is a local minimum value; and

iv. $(D(2,-1) = -36$, so $h(2,-1) = 3$ is neither a local minimum value nor a local maximum value.

6. ACME Widgets produces two competing products, type A widgets and type B widgets. The joint demand functions for these products are

$$Q_A = 100 - 3P_A + 2P_B \text{ and } Q_B = 60 + 2P_A - 2P_B$$

and ACME's cost function is

$$C = 20Q_A + 30Q_B + 1200.$$

Find the prices that ACME should charge to maximize their profit, the corresponding output levels and the max profit. Justify your claim that the prices you found yield the absolute maximum profit.

ACME's profit function is

$$\begin{aligned}
\Pi &= P_A Q_A + P_B Q_B - C \\
&= 100P_A - 3P_A^2 + 2P_A P_B + 60P_B + 2P_A P_B - 2P_B^2 \\
&\qquad - \left[20(100 - 3P_A + 2P_B) + 30(60 + 2P_A - 2P_B) + 1200 \right] \\
&= -3P_A^2 + 4P_A P_B - 2P_B^2 + 100P_A + 80P_B - 5000.
\end{aligned}$$

(i) Critical point(s):

$$\left.\begin{aligned}
\Pi_{P_A} &= -6P_A + 4P_B + 100 = 0 \\
\Pi_{P_B} &= 4P_A - 4P_B + 80 = 0
\end{aligned}\right\}$$

Now, $\Pi_{P_B} = 0 \Longrightarrow 4P_B = 4P_A + 80$, and plugging this into the first equation gives

$$-6P_A + (4P_A + 80) + 100 = 0 \Longrightarrow -2P_A + 180 = 0 \Longrightarrow P_A = 90 \Longrightarrow P_B = 110.$$

So there is only one critical point, and the critical prices are $P_A = 90$ and $P_B = 110$, with corresponding output/demand levels $Q_A = 50$ and $Q_B = 20$.

(ii) Second derivative test:

$$\left.\begin{aligned}
\Pi_{P_A P_A} &= -6 \\
\Pi_{P_B P_B} &= -4 \\
\Pi_{P_A P_B} &= 4
\end{aligned}\right\} \Longrightarrow D = (-6) \cdot (-4) - 16 = 8 > 0.$$

Since $D > 0$ and $\Pi_{P_A P_A} = -6 < 0$, and the second derivatives are all constant, it follows that $\Pi(90, 110) = 3900$ is the absolute maximum profit.

7. An electronics retailer has determined that the number N of laptops she can sell per week is

$$N = \frac{9x}{4+x} + \frac{20y}{5+y}$$

where x is her weekly expenditure on radio advertising and y is her weekly expenditure on internet advertising, both measured in \$100s. Her weekly profit is \$400 per sale, less the cost of advertising.

Find the amount of money that the retailer should spend on radio and internet advertising, respectively, to maximize her weekly profit. Verify that the point you found yields a relative maximum value. What is the maximum profit?

The first step is to find the weekly profit function, P, which is \$400 times the number of laptops sold minus the cost of advertising:

$$P = 400N - 100(x+y) = 100\left(\frac{36x}{4+x} + \frac{80y}{5+y} - x - y\right).$$

Next, first-order conditions:

$$\begin{aligned}
P_x = 0 \quad &\Longrightarrow \quad 100\left(\frac{144}{(4+x)^2} - 1\right) = 0 \Longrightarrow \frac{144}{(4+x)^2} = 1 \Longrightarrow (4+x)^2 = 144 \\
P_y = 0 \quad &\Longrightarrow \quad 100\left(\frac{400}{(5+y)^2} - 1\right) = 0 \Longrightarrow \frac{400}{(5+y)^2} = 1 \Longrightarrow (5+y)^2 = 400
\end{aligned}$$

It follows that $4 + x = \pm 12$ and $5 + y = \pm 20$, and since both x and y must be nonnegative, we conclude that the critical numbers are $x^* = 12 - 4 = 8$ and $y^* = 20 - 5 = 15$.

The corresponding profit is

$$P^* = 100\left(\frac{288}{12} + \frac{1200}{20} - 8 - 15\right) = 6100.$$

To verify that this is the maximum profit, we use the second derivative test:

$$P_{xx} = -\frac{288}{(4+x)^3}, \; P_{yy} = -\frac{800}{(5+y)^3} \; \text{ and } \; P_{xy} = 0,$$

so

$$D(x,y) = \frac{288 \cdot 800}{(4+x)^3(5+y)^3} \Longrightarrow D(8,15) = \frac{288 \cdot 800}{12^3 \cdot 20^3} > 0$$

and

$$P_{xx}(8,15) = -\frac{288}{12^3} < 0$$

which shows that P^* is a maximum.