Study Guide 5

Optimization, constrained optimization and applications of integrals.

(*) *In all the* ***constrained*** *optimization problems below, you may assume that the critical values you find are in fact the optimal values you seek.*

1. Compute the producers' and consumers' surplus at equilibrium for the market with the supply and demand equations:

$$\text{supply:} \quad p = 0.1q + 5; \qquad \text{demand:} \quad p = 40 - \frac{q}{10} - \frac{q^2}{100}.$$

2. A monopolistic firm sells one product in two markets, A and B. The daily demand equations for the firm's product in these markets are given by

$$Q_A = 100 - 0.4P_A \qquad \text{and} \qquad Q_B = 120 - 0.5P_B,$$

where Q_A and Q_B are the demands and P_A and P_B are the prices for the firm's product in markets A and B, respectively. The firm's constant marginal cost is \$40 and the its daily fixed cost is \$2500.

a. Find the prices that the firm should charge in each market to maximize its daily profit. Use the second derivative test to verify that the prices you found yield the *absolute* maximum profit.

b. Use the *envelope theorem* (and linear approximation) to estimate the change in the Firm's max profit if the marginal cost of their product increases to \$40.75.

3. Jack's (gustatory) utility function is

$$U(x, y, z) = 5 \ln x + 7 \ln y + 18 \ln z,$$

where x is the number of fast-food meals Jack consumes in a month; y is the number of 'diner' meals he consumes in a month; and z is the number of 'fancy restaurant' meals he consumes in a month.

The average price of a fast-food meal is $p_x = \$4.00$; the average price of a 'diner' meal is $p_y = \$8.00$; and the average price of a 'fancy restaurant' meal is $p_z = \$30.00$.

a. How many meals of each type should Jack consumer per month to maximize his utility, if his monthly budget for these meals is $\beta = \$1200.00$?

b. By approximately how much will Jack's utility increase if his budget increases by \$50.00? Explain your answer.

4. A firm's production function is given by

$$Q = 10K^{0.4}L^{0.7},$$

where Q is the firm's annual output, K is the annual capital input, and L is the annual labor input. The cost per unit of capital is \$1000, and the cost per unit of labor is \$4000.

a. Find the levels of labor and capital inputs that **minimize** the cost of producing an output of $Q = 20,000$ units.

b. Find the levels of labor and capital inputs that **minimize** the cost of producing an output of $Q = q$ units. Express your answer in terms of q.

5. The production function for ACME Widgets is

$$Q = 2k^2 + kl + 5l^2,$$

where k and l are the numbers of units of capital and labor input, respectively, and Q is their output, *measured in 1000s of widgets.* The price per unit of capital input is $p_k = \$1000$ and the price per unit of labor input is $p_l = \$2500$.

a. How many units of capital and labor input should ACME use to ***minimize the cost*** of producing 65000 widgets? What is the *average cost per widget*?

b. By approximately how much will ACME's cost rise if they raise their output from 65000 widgets to 65500 widgets? Justify your answer.

c. By approximately how much will ACME's minimum cost increase (from part a.) if the cost per unit of capital increases to \$1100? Use the *envelope theorem* and linear approximation.

6. The annual output for a luxury hotel chain is given by $Q = 30K^{2/5}L^{1/2}R^{1/4}$, where K, L and R are the capital, labor and real estate inputs, all measured in \$1,000,000 s, and Q is the average number of rooms rented per day.

The hotel chain's annual budget is $B = \$69$ million.

a. How should they allocate this budget to the three inputs in order to *maximize* their annual output? What is the maximum output?

b. What is the critical value of the multiplier when output is maximized?

c. Use your answer to **b.** to compute the *approximate* change in the firm's maximum output if their annual budget increases by \$500,000? Explain your answer.