SN 2 — Solutions to Exercises

1.
- $f(1,1) = 0$;
- $f_x = 3x^2y + 6xy^2 - 2y^2 + 3$ so $f_x(1,1) = 10$;
- $f_y = x^3 + 6x^2y - 4xy - 5$ so $f_y(1,1) = -2$;
- $f_{xx} = 6xy + 6y^2$ so $f_{xx}(1,1) = 12$;
- $f_{xy} = 3x^2 + 12xy - 4y$ so $f_{xy}(1,1) = 11$;
- $f_{yy} = 6x^2 - 4x$ so $f_{yy}(1,1) = 2$.

This means that the Taylor polynomial for f centered at $(1,1)$ is

$$T(x,y) = 10(x-1) - 2(y-1) + 6(x-1)^2 + 11(x-1)(y-1) + (y-1)^2.$$

2.
- $Q(1,8) = 20$;
- $Q_K = (20/3)K^{-1/3}L^{1/3}$ so $Q_K(1,8) = 40/3$;
- $Q_L = (10/3)K^{2/3}L^{-2/3}$ so $Q_(1,8) = 5/6$;
- $Q_{KK} = -(20/9)K^{-4/3}L^{1/3}$ so $Q_{KK}(1,8) = -40/9$;
- $Q_{KL} = (20/9)K^{-1/3}L^{-2/3}$ so $Q_{KL}(1,8) = 5/9$;
- $Q_{LL} = -(20/9)K^{2/3}L^{-5/3}$ so $Q_{LL}(1,8) = -5/72$.

This means that the Taylor polynomial for Q centered at $(1,8)$ is

$$T(K,L) = 20 + \frac{40}{3}(K-1) + \frac{5}{6}(L-8) - \frac{20}{9}(K-1)^2 + \frac{5}{9}(K-1)(L-8) - \frac{5}{144}(L-8)^2,$$

and

$$Q(1.5, 8.4) \approx T(1.5, 8.4) = 20 + \frac{10}{3} + \frac{1}{3} - \frac{5}{9} + \frac{1}{9} - \frac{1}{180} = \frac{4179}{180} \approx 23.21667.$$

3.
- $g(0,0) = 0$;
- $g_u = (2 + 4u^2 + 6uv)e^{u^2+v^2}$ so $g_u(0,0) = 2$;
- $g_v = (3 + 6uv + 9v^2)e^{u^2+v^2}$ so $g_v(0,0) = 3$;
- $g_{uu} = (12u + 6v + 8u^3 + 12u^2v)e^{u^2+v^2}$ so $g_{uu}(0,0) = 0$;
- $g_{uv} = (6u + 4v + 8u^2v + 12uv^2)e^{u^2+v^2}$ so $g_{uv}(0,0) = 0$;
- $g_{vv} = (6u + 24v + 12uv^2 + 18v^3)e^{u^2+v^2}$ so $g_{vv}(0,0) = 0$;

Thus, the *Quadratic* Taylor polynomial for g centered at $(0,0)$ is given by $T(x,y) = 2x + 3y$ (a *linear* function).

4.
- $H(3,2) = 4$;
- $H_x = (2x + 5y)^{-1/2}$; so $H_x(3,2) = \frac{1}{4}$;
- $H_y = 2.5(2x + 5y)^{-1/2}$ so $H_y(3,2) = \frac{5}{8}$;
- $H_{xx} = -(2x + 5y)^{-3/2}$ so $H_x(3,2) = -\frac{1}{64}$;
- $H_{xy} = -2.5(2x + 5y)^{-3/2}$ so $H_{xy}(3,2) = -\frac{5}{128}$;

- $H_{yy} = -\frac{25}{4}(2x+5y)^{-3/2}$ so $H_{yy}(3,2) = -\frac{25}{256}$.

This means that the Taylor polynomial for $H(x,y)$ centered at $(3,2)$ is

$$T(x,y) = 4 + \frac{1}{4}(x-3) + \frac{5}{8}(y-2) - \frac{1}{128}(x-3)^2 - \frac{5}{128}(x-3)(y-2) - \frac{25}{512}(y-2)^2,$$

and

$$\sqrt{17} = H(3.25, 2.1) \approx T(3.25, 2.1) = 4 + \frac{1}{16} + \frac{1}{16} - \frac{1}{2048} - \frac{1}{1024} - \frac{1}{2048} = \frac{2111}{512} \quad (\approx 4.123).$$

5. Compute $T_2(x,y,z)$ for the function

$$g(x,y,z) = 2\ln x + 4\ln y - z(5x+8y-60),$$

centered at the point $(x_0, y_0, z_0) = (4, 5, 0.1)$.

- $g(4,5,0.1) = 2\ln 4 + 4\ln 5 - 0.1(0) = \ln(4^2 \cdot 5^4) = \ln 10000$ (≈ 9.21)
- $g_x = \frac{2}{x} - 5z$; so $g_x(4,5,0.1) = \frac{2}{4} - 0.5 = 0$
- $g_y = \frac{4}{y} - 8z$; so $g_y(4,5,0.1) = \frac{4}{5} - 0.8 = 0$
- $g_z = -(5x+8y-60)$; so $g_z(4,5,0.1) = -(20+40-60) = 0$
- $g_{xx} = -\frac{2}{x^2}$ so $g_{xx}(4,5,0.1) = -\frac{1}{16}$
- $g_{xy} = 0$; (so $g_{xy}(4,5,0.1) = 0$)
- $g_{xz} = -5$; (so $g_{xz}(4,5,0.1) = -5$)
- $g_{yy} = -\frac{4}{y^2}$; so $g_{yy}(4,5,0.1) = -\frac{4}{25}$
- $g_{yz} = -8$; (so $g_{yz}(4,5,0.1) = -8$)
- $g_{zz} = 0$; (so $g_{zz}(4,5,0.1) = 0$)

Therefore the quadratic Taylor polynomial for $g(x,y,z)$ centered at $(4,5,0.1)$ is

$$T_2(x,y,z) = \ln 10000 - \frac{1}{32}(x-4)^2 - 5(x-4)(z-0.1) - \frac{2}{25}(y-5)^2 - 8(y-5)(z-0.1)$$