Goal:

Describe succinct and precise notation for expressions like

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21$$

or

$$f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n).$$

Why?

Because good notation (language) ultimately makes it easier to understand and use mathematics.

Summation Notation

$$A_m + A_{m+1} + A_{m+2} + \cdots + A_n = \sum_{k=m}^{n} A_k .$$

- k is called the ***index of summation***,
- m is the ***lower limit of summation***, and
- n is the ***upper limit of summation***.

(*) We require that $n \geq m$

(*) The index of summation k starts at m and ***increases by steps of size*** 1 until it reaches n.

(*) Most commonly, $m = 0$ or $m = 1$.

The **terms** in the sum, A_m, A_{m+1}, A_{m+2}, $\ldots$, A_n, are typically either

- **A list of values** — e.g., a list of grades — in which case the index of summation is simply *enumerating* these values as first, second, third, etc. This is the situation that you will encounter in statistics and econometrics, for example.

  Or

- **Functions of the index of summation**, e.g.,

$$f(1) + f(2) + \cdots + f(n) \;=\; \sum_{k=1}^{n} f(k).$$

  This is the case we will be considering here.

Examples:

$$1 + 2 + 3 + \cdots + 20 = \sum_{k=1}^{20} k.$$

$$1 + 4 + 9 + \cdots + 100 = \sum_{l=1}^{10} l^2.$$

$$1 + 3 + 5 + \cdots + 99 = \sum_{j=1}^{50} (2j - 1).$$

(*) There's no rule that the index of summation has to be a k. Other common choices are i, j, l, m and n.

Properties:

Since 'summation' is just another word for 'addition', the basic properties of addition hold:

1. Distributivity:

$$\begin{aligned}\sum_{k=m}^{n}(C\cdot f(k)) &= Cf(m)+Cf(m+1)+\cdots+Cf(n)\\ &= C(f(m)+f(m+1)+\cdots+f(n))\\ &= C\cdot\left(\sum_{k=m}^{n} f(k)\right).\end{aligned}$$

2. Commutativity and Associativity:

$$\sum_{k=m}^{n}\big(\,g(k)\pm f(k)\big)=\left(\sum_{k=m}^{n} g(k)\right)\pm\left(\sum_{k=m}^{n} f(k)\right).$$

Formulas:

0. $$\sum_{k=1}^{n} C = \overbrace{C + C + \cdots + C}^{n} = nC.$$

1. $$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n^2 + n}{2} \text{ ... why?}$$

Quick explanation:

Write $S_n = 1 + 2 + 3 + \cdots + n$,

then it is also true that $S_n = n + (n-1) + (n-2) + \cdots + 1$.

Therefore

$$\begin{array}{rclcccccccccc} S_n & = & & 1 & + & 2 & + & \cdots & + & n & & \\ +S_n & = & + & n & + & n-1 & + & \cdots & + & 1 & & \\ \hline 2S_n & = & & (n+1) & + & (n+1) & + & \cdots & + & (n+1) & = & n(n+1) \end{array}$$

so $S_n = \dfrac{1}{2} n(n+1) = \dfrac{n^2 + n}{2}$

Different (better?) explanation:

Observation: $(k+1)^2 = k^2 + 2k + 1,$ so

$$(k+1)^2 - k^2 = 2k + 1.$$

Therefore

$$\begin{aligned}\sum_{k=1}^{n} \left[(k+1)^2 - k^2\right] &= \sum_{k=1}^{n}(2k+1) \\ &= \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1 \\ &= 2\left(\sum_{k=1}^{n} k\right) + n.\end{aligned}$$

On the other hand...

$$\begin{aligned}\sum_{k=1}^{n} \left[(k+1)^2 - k^2\right] &= \left[2^2 - 1^2\right] + \left[3^2 - 2^2\right] + \cdots + \left[(n+1)^2 - n^2\right] \\ &= (n+1)^2 - 1 = n^2 + 2n.\end{aligned}$$

This means that

$$2\left(\sum_{k=1}^{n} k\right) + n = n^2 + 2n,$$

so

$$\sum_{k=1}^{n} k = \frac{1}{2}\left(n^2 + 2n - n\right) = \frac{n^2+n}{2} = \frac{1}{2}n^2 + \frac{1}{2}n.$$

(*) This explanation is better because the same approach can be used to derive formulas for sums of squares, sums of cubes, sums of fourth powers, etc.

2. $$\sum_{k=1}^{n} k^2 = \frac{2n^3+3n^2+n}{6} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

3. $$\sum_{k=1}^{n} k^3 = \frac{n^4+2n^3+n^2}{4} = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2.$$

***m*.** More generally: $$\sum_{k=1}^{n} k^m = \frac{1}{m+1}n^{m+1} + P_m(n),$$

where P_m is a polynomial of degree m.